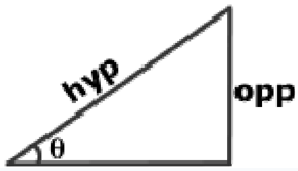
Trigonometric ratios: Sine

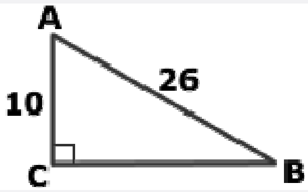
Right triangles have ratios to represent the angles formed by the hypotenuse and its legs. There are six basic trigonometric functions, which are tabulated here along with equations relating them to one another: Sine, cosine, tangent, cosecant, secant and cotangent.

Sine: The sine of an angle in a right triangle equals the opposite side divided by the hypotenuse.

$$\sin\theta = \frac{opp}{hyp}$$



Find Sin A and Sin B



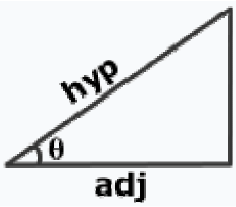
Using the Pythagorean Theorem, the missing leg must be 24, then

$\sin A = \frac{24}{26} = \frac{12}{13}$ and $\sin B = \frac{10}{26} = \frac{5}{13}$

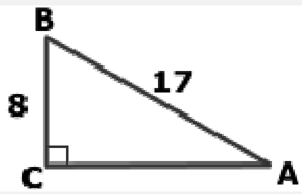
Trigonometric ratios: Cosine

Cosine: The cosine of an angle in a right triangle equals the adjacent side divided by the hypotenuse.

$$\cos\theta = \frac{adj}{hyp}$$



Find Cos A and Cos B



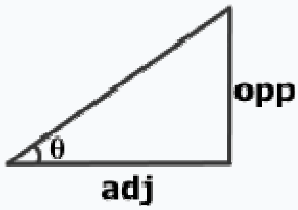
Using the Pythagorean Theorem, the missing leg must be 15, then

$\text{Cos } A = \frac{15}{17}$ and $\text{Cos } B = \frac{8}{17}$

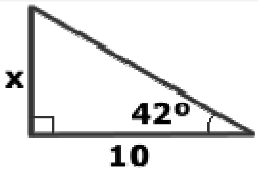
Trigonometric ratios: Tangent

Tangent: The tangent is the opposite side divided by the adjacent side.

$$\tan\theta = \frac{\text{opp}}{\text{adj}}$$



Find x



$$\tan 42^{o} = \frac{x}{10}$$

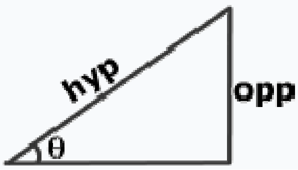
x=10·tan42º
x=9

The reciprocal trigonometric ratios: cosecant

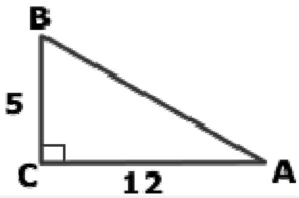
It is said that there are six ratios possible for the lengths of the sides of a right triangle. You have learned about the sine (sin), cosine (cos), and tangent (tan) ratios. The three other trigonometric ratios are their reciprocals.

Cosecant (csc): is the reciprocal of sine, i.e. the ratio of the length of the hypotenuse to the length of the opposite side.

$$\csc\theta = \frac{1}{\sin\theta} = \frac{\text{hyp}}{\text{opp}}$$



Find the value of cscB



First find the length of the hypotenuse. Using the Pythagorean Theorem, or recognizing the Pythagorean Triple 5-12-13, the length of the hypotenuse is 13.

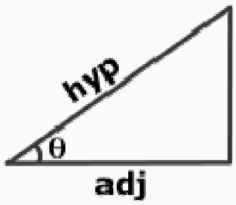
$$\csc B = \frac{13}{12}$$

The reciprocal trigonometric ratios: secant

It is said that there are six ratios possible for the lengths of the sides of a right triangle. You have learned about the sine (sin), cosine (cos), and tangent (tan) ratios. The three other trigonometric ratios are their reciprocals.

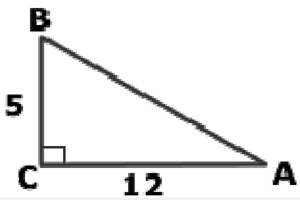
Secant (sec): is the reciprocal of cosine, i.e. the ratio of the length of the hypotenuse to the length of the adjacent side.

$$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{hyp}}{\text{adj}}$$



Important note: There is a big difference between sec θ and $\cos^{-1}x$. The first one means "1/cos θ". The second one involves finding an **angle** whose cosine is *x*. So on your calculator, don't use your $\cos^{-1}$ button to find sec θ.

Find the value of secB



First find the length of the hypotenuse. Using the Pythagorean Theorem, or recognizing the Pythagorean Triple 5-12-13, the length of the hypotenuse is 13.

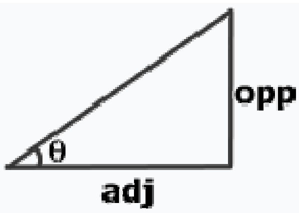
$$\sec B = \frac{13}{5}$$

The reciprocal trigonometric ratios: cotangent

It is said that there are six ratios possible for the lengths of the sides of a right triangle. You have learned about the sine (sin), cosine (cos), and tangent (tan) ratios. The three other trigonometric ratios are their reciprocals.

Cotangent (cot*):* is the reciprocal of tan, i.e. the ratio of the length of the adjacent side to the length of the opposite side.

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\text{adj}}{\text{opp}}$$

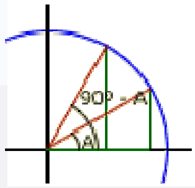


Important note: There is a big difference between cot θ and $\tan^{-1}x$. The first one means "1/tan θ". The second one involves finding an **angle** whose tangent is *x*. So on your calculator, don't use your $\tan^{-1}$button to find cot θ.

Trigonometric functions at related angles

Using the geometric symmetry of the unit circle, some trigonometric functions can be established. You can calculate the trigonometric functions of an angle in the second, third or fourth quadrant using its ratio with the first quadrant.

Complementary angles.
Two angles are complementary if they **add up to 90 degrees.**

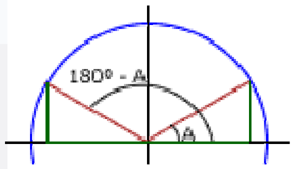


If A and B are two angles where A+B=90º, that is, B=90º - A, we have:
$sinA = cos\ B$, so that, $sin\ A = cos(90^o\text{-}A)$
$cosA = sin\ B$, so that, $cosA=sin(90^o\text{-}A)$
Similarly, $tanA = cotB$

Trigonometric functions at related angles

Using the geometric symmetry of the unit circle, some trigonometric functions can be established. You can calculate the trigonometric functions of an angle in the second, third or fourth quadrant using its ratio with the first quadrant.

Supplementary angles.
Two angles are supplementary if they **add up to 180 degrees.**



If A and B are two angles where A+B=180º , that is, B=180º - A, we have:
$sinA = sin\ B$, so that, $sin\ A = sin(180^o\text{-}A)$
$cosA = \text{-}cos\ B$, so that, $cosA = \text{-}cos(180^o\text{-}A)$
Similarly $tanA = \text{-}tanB$

Using these formulas, you can calculate the trigonometric functions of an angle in the second quadrant if you know the trigonometric functions of its supplementary angle.

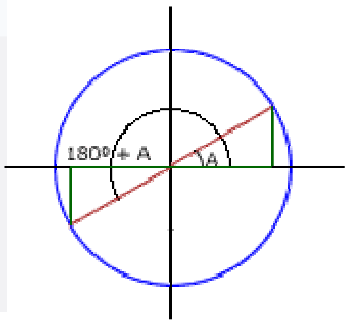
Trigonometric functions at related angles

Using the geometric symmetry of the unit circle, some trigonometric functions can be established. You can calculate the trigonometric functions of an angle in the second, third or fourth quadrant using its ratio with the first quadrant.

Angles that differ by 180º

If A and B are two angles that B - A= 180º, that is, B=180º + A, then:
$sinA = \text{-}sin\ B$, so that, $sin\ A = \text{-}sin(180^o\text{+}A)$
$cosA = \text{-}cos\ B$, so that, $cosA = \text{-}cos(180^o\text{+}A)$
Similarly $tanA = tanB$

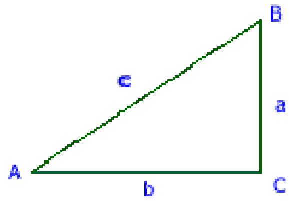
Using these formulas, you can calculate the trigonometric functions of an angle in the third quadrant if you know the trigonometric functions of the angle that differs with it 180º.



Trigonometric functions at related angles

Using the geometric symmetry of the unit circle, some trigonometric functions can be established. You can calculate the trigonometric functions of an angle in the second, third or fourth quadrant using its ratio with the first quadrant.

Opposite angles.
Two angles are opposite angles if they **add up to 0º or a multiple of 360º.**



If A and B are two angles that B - A= k·360º where k is an integer, then:
$\sin A = -\sin B$, so that, $\sin A = -\sin(-A)$
$\cos A = \cos B$, so that, $\cos A = \cos(-A)$
Similarly $\tan A = -\tan B$

Using these formulas, you can calculate the trigonometric functions of an angle in the fourth quadrant if you know the trigonometric functions of its opposite angle.

Trigonometric identities

The following equations are the most basic and important trigonometric identities. These equations are true for any angle. From them, countless additional identities can be formed.

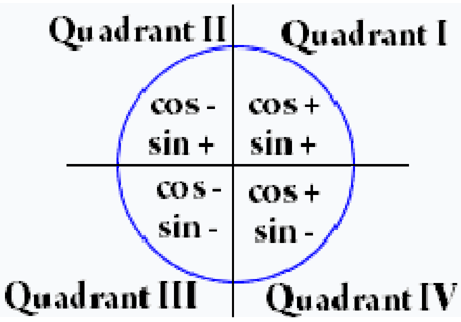
Basic trigonometric identities:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

Pythagorean Identities:
a) $\sin^2 A + \cos^2 A = 1$
b) $1 + tg^2 A = \sec^2 A$
c) $1 + \cot^2 A = \csc^2 A$

You can use the trigonometric identities given above to find trigonometric ratios if you are given the quadrant an angle lies in and the value of one trigonometric ratio.

It will be useful to remember the sign of the trigonometric ratios in each quadrant:



If $\sin\theta = \frac{3}{5}$ and $\frac{\pi}{2} < \theta < \pi$, find $\cos\theta$ and $\tan\theta$.

$\cos^2\theta = 1 - \sin^2\theta = 1 - (\frac{3}{5})^2 = \frac{16}{25}$, then $\cos\theta = -\frac{4}{5}$ because $\frac{\pi}{2} < \theta < \pi$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = -\frac{3}{4}$$

Solving right triangles

To **solve** a triangle is equivalent to find unknown parts in terms of known parts.

We can use the Pythagorean theorem and properties of sines, cosines, and tangents to solve the triangle.

Solving right triangles when we know two sides.
1. The Pythagorean theorem will give us the unknown side.
2. Use a sine, cosine, or tangent to determine an unknown angle.
3. Remember that the sum of the three angles equals 180º to find the third missing angle.

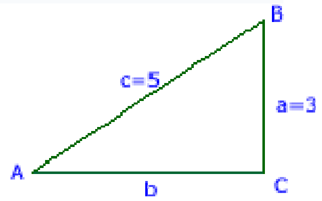
Solve the triangle showed:

1. Using the Pythagorean theorem:
$c^2 = a^2 + b^2$, por tanto $b^2 = c^2 - a^2$;
$b^2 = 5^2 - 3^2$; $b^2 = 25 - 9$; $b^2 = 16$; $b = 4$.

2. Using the sine:
senA=3/5, then A=arcsen(3/5) ; A= 36'87º.

3. A+B+C=180º and C=90º, then B=180º -90º-36'87º ; B= 53'13º

So that: The sides are 3, 4 and 5, and the angles are 90º, 36'87º y 53'13º.



Solving right triangles

To **solve** a triangle is equivalent to find unknown parts in terms of known parts.

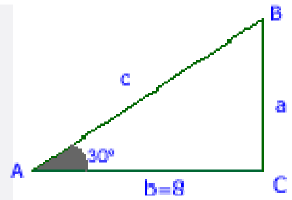
We can use the Pythagorean theorem and properties of sines, cosines, and tangents to solve the triangle.

Solving right triangles when we know one side and one angle.

1. Use a sine, a cosine or a tangent to find another side (whether you use a sine, cosine, or tangent depends on which side and angle you know).
2. The Pythagorean theorem will give us the unknown side.
3. Remember that the sum of the three angles equals 180º to find the third missing angle.

Solve the triangle showed:
1. b=8 and cos30º=8/c, then c=8/cos30º ; c= 9'24.
2. Using Pythagorean theorem a=4'62.
3. A=30º and C=90º, then B=180º-90º-30º ; B=60º.

The sides are 9'24, 8, 4'24 and the angles are 30º, 60º y 90º.



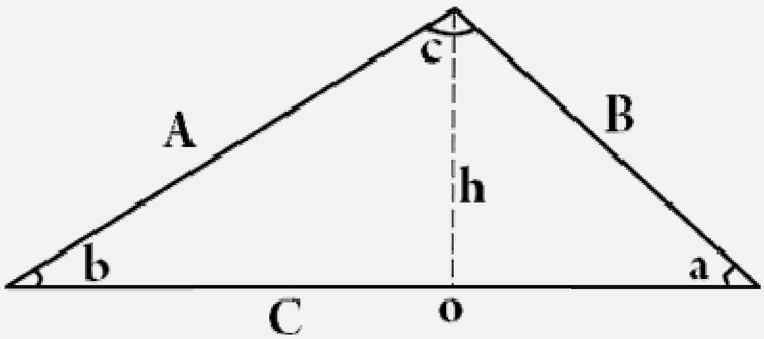
Law of sines

The Law of Sines is the relationship between sides and angles in any triangle.

The sides of a triangle are to one another in the same ratio as the sines of their opposite angles.

Let's see...

Consider this triangle:



Then, the Law of Sines states that:

$$\frac{A}{\sin a} = \frac{B}{\sin b} = \frac{C}{\sin c}$$

The law can also be written as the reciprocal:

$$\frac{\sin a}{A} = \frac{\sin b}{B} = \frac{\sin c}{C}$$

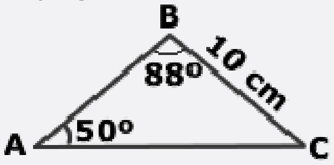
Proof:

The perpendicular, *oc*, splits this triangle into two right-angled triangles. This lets us calculate *h* in two different ways

- Using the triangle cao gives $h = B \sin a$
- Using the triangle cbo gives $h = A \sin b$
- Eliminate h from these two equations $A \sin b = B \sin a$
- Rearrange $\dfrac{A}{\sin a} = \dfrac{B}{\sin b}$

By using the other two perpendiculars the full law of sines can be proved. **QED.**

Find AC.

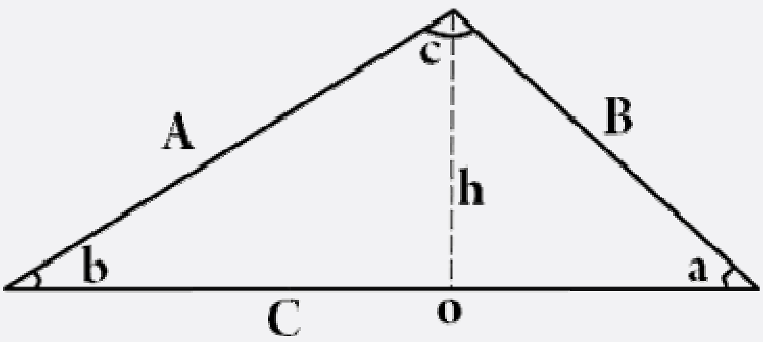


$\frac{\sin 50}{10} = \frac{\sin 88}{AC}$; AC·sin50=10·sin88; AC= $\frac{10\sin 88;}{\sin 50}$; AC=13 cm

The law of cosines

The Law of Cosines is a extension of the Pythagorean theorem to use when triangles that are not right-angled.

Consider this triangle:



Then, the Law of Cosines states that:

$$A^2 = B^2 + C^2 - 2BC \cdot \cos a$$
$$B^2 = A^2 + C^2 - 2AC \cdot \cos b$$
$$C^2 = A^2 + B^2 - 2AB \cdot \cos c$$

Proof:

The perpendicular, *oc*, divides this triangle into two right angled triangles, *aco* and *bco*.

First we will find the length of the other two sides of triangle *aco* in terms of known quantities, using triangle *bco*.

$h = A \sin b$

Side C is split into two segments, total length *C*.

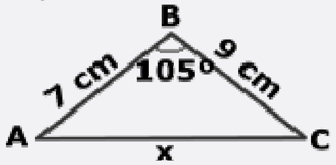
ob, length $A \cos b$
ao, length $C - A \cos b$

Now we can use Pythagoras to find *B*, since $B^2 = ao^2 + h^2$

$$\begin{aligned} B^2 &= (C - A\cos b)^2 + A^2 \sin^2 b \\ &= C^2 - 2AC\cos b + A^2\cos^2 b + A^2 \sin^2 b \\ &= A^2 + C^2 - 2AC\cos b \end{aligned}$$

The corresponding expressions for *A* and *C* can be proved similarly.

Find x.



$c^2=a^2+b^2-2ac\cos C$

$x^2=7^2+9^2-2\cdot 7\cdot 9\cdot \cos 105º=162.6$

$x=12.8$ cm